Bond Market Clienteles, the Yield Curve and the Optimal Maturity Structure of Government Debt

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Abstract

This paper provides a clientele-based model of the yield curve and develops a novel theory of the optimal maturity composition of government debt. We consider an infinite-horizon stochastic overlapping generations economy in which agents live for three periods. Clienteles correspond to generations of agents at different stages of their life cycle. We examine how changes in the clientele mix and the maturity structure of government debt affect the equilibrium yield curve. We show that an optimal maturity structure exists even in the absence of distortionary taxes. The government can use the maturity structure to achieve efficient intergenerational risksharing by effectively trading on behalf of future generations. The optimal fraction of long-term debt increases in the weight of the long-horizon clientele, provided that agents are more risk-averse than log.

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1 Introduction

The government-bond market involves many distinct investor clienteles. For example, pension funds and insurance companies invest typically in long maturities as a way to hedge their long-term liabilities, while asset managers and banks’ treasury departments hold shorter maturities. Clientele’s demands vary over time in response to demographic or regulatory changes. This time-variation can have important effects both on the yield curve and on the government’s debt-issuance policy.

The UK pension reform provides a stark illustration of clientele effects. Starting in 2005, pension funds were required to mark their liabilities to market. This caused them to tilt their assets towards longer-maturity bonds and had a dramatic impact on the yield curve. For example, in January 2006 the inflation-indexed bond maturing in 2011 was yielding 1.5%, while the 2055 bond was yielding only 0.6%. The 0.6% yield is very low relative to the 2.3% historical average of UK long real rates.\(^1\) Moreover, the downward-sloping yield curve is hard to attribute to an expectation of rates dropping below 0.6% after 2011. In turn, the steep decline in long rates induced the UK Treasury to tilt debt issuance towards long maturities. For example, bonds with maturities of fifteen years or longer constituted 58% of issuance during financial year 2006-7, compared to an average of 40% over the previous four years.\(^2\)

While clientele considerations seem to influence the yield curve and the practice of government debt issuance, they are largely absent from theoretical models. In this paper, we provide a model of government bond markets that emphasizes the role of clienteles. In our setting clienteles are most naturally interpreted as different demographic groups. More specifically, we consider an overlapping generations (OLG) economy where different generations of agents are at different stages of their life cycle. Clienteles for bonds of different maturities arise because distinct cohorts differ in their investment horizon. We have three types of results. First, we derive general effects of changes in demand and supply on the yield curve for arbitrary maturity structures of government debt. In particular, we show that clientele-driven changes in demand as well as exogenous changes in the maturity composition of government debt have intuitive effects on the equilibrium yield curve for plausible parameter values. Our second set of results is normative in nature. In a stochastic OLG setting the maturity structure of government debt is non-neutral even when taxes are non-distortionary because it affects intergenerational risksharing. We show that there is an optimal maturity structure in our framework. The optimal maturity structure achieves ex ante efficient intergenerational risksharing by effectively allowing the government to trade on behalf of unborn

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\(^1\)Source: Dimson, Marsh, and Staunton (2002).

\(^2\)The issuance numbers are from the website of the UK Debt Management Office. For a more detailed account of the UK pension reform, see Greenwood and Vayanos (2010). Another illustration of catering to clienteles is the French Treasury’s first-time issuance of a 50-year bond in 2005, in response to strong demand by pension funds.
generations. Finally, we examine the main features of the optimal debt policy. In particular, we show how the optimal maturity structure relates to the clientele mix. We also derive the equilibrium term structure under the optimal policy and examine the connection between welfare maximization and funding cost minimization, finding some interesting parallels.

We conduct our analysis in an infinite horizon OLG setting, described in Section 2. Agents live for three periods and they have CRRA preferences. Each generation receives an exogenous endowment in the first period of life, invests for two periods and consumes in the final period. All agents have access to a one-period riskless linear production technology. The return on this technology pins down the one-period riskfree interest rate. For tractability, we focus on a simple stochastic structure: the technological rate of return is known and constant over time, except in period $j$ where it can take a high or a low value. This can be interpreted as an aggregate productivity shock. Uncertainty is resolved in period $j$. We assume the government can issue or invest in non-contingent, zero-coupon bonds of different maturities and levy nondistortionary taxes on agents’ endowments.

Section 3 studies demand and supply effects for a general maturity structure of government debt. First, we analyze how changes in the clientele mix affect the equilibrium yield curve. Generations born before period $j - 2$ do not live long enough to see the realization of uncertainty and there is no motive for them to trade in our framework. We interpret the generations born in periods $j - 2$ and $j - 1$ respectively as the short- and long-horizon clienteles in period $j - 1$ and we control the clientele mix by altering their relative wealth. We show that for a coefficient of relative risk aversion higher than one, an increase in the relative wealth of the long-horizon clientele translates into a lower two-period interest rate. Second, we show that lengthening the maturity structure raises the slope of the yield curve. The intuition is that any incremental bond issuance must be absorbed by the generations that are present in the market in period $j - 1$. These agents, however, do not experience offsetting changes in future tax rates and therefore their consumption allocation is sensitive to changes in the maturity structure. Lengthening the maturity structure shifts consumption of the generations born in periods $j - 2$ and $j - 1$ towards the state where interest rates are low, which lowers the valuation of two-period bonds by both generations, raises the two-period interest rate and the slope of the yield curve. While this mechanism is consistent with practical intuition, it cannot arise in representative agent models commonly used in the literature.

Section 4 introduces a clientele-based theory of the optimal maturity structure. It is well known that in an OLG environment markets are fundamentally incomplete and intergenerational risksharing can be inefficient because future generations cannot trade before they are born. In this context, the government can improve risksharing through its debt-issuance policy. In our model, adjusting the mix of one- and two-period debt in period $j - 1$ provides a simple way of redistributing consumption across generations state by state. For example, as suggested above, lengthening the
maturity structure redistributes consumption away from generations born in period $j$ and later towards the generations born in periods $j - 2$ and $j - 1$ in the state where the interest rate is low.\(^3\) We show that an optimal maturity structure of government debt exists in our model and derive it in two steps. First, we consider a hypothetical case where all generations can trade in one and two-period bonds before the realization of the shock in period $j$. In this benchmark case, markets are complete and the equilibrium is ex ante efficient.\(^4\) Then we show how the government can implement the efficient allocation by an appropriate choice of the maturity structure of its debt. The key insight is that in order to replicate the complete-markets allocation, the government issues the quantity of two-period bonds that all generations born in period $j$ or later would sell on aggregate if allowed to trade in period $j - 1$ in an economy where two-period bonds are in zero net supply. In other words, the government’s issuance policy replicates the actions of private agents not present in the market.

Section 5 analyzes the main features of optimal debt policy and the associated equilibrium yield curve. We first characterize the optimal supply of long-term debt by the government as a function of preferences and endowments. We also analyze the comparative statics of changes in the clientele mix and their effect on the optimal maturity structure. This type of analysis cannot be undertaken in a representative agent model. Intuitively the optimal maturity structure should involve more short-term debt when the short-horizon clientele is wealthier. Our model suggests that this intuitive effect only prevails when agents’ coefficient of relative risk aversion $\gamma$ is strictly larger than one. Asset-pricing research (e.g., equity premium puzzle) generally supports the assumption $\gamma > 1$. Since clientele effects seem prevalent in practice, our results generate a similar conclusion in the context of the term structure.\(^5\) Next, we investigate excess returns on long-term bonds and the slope of the yield curve under the optimal policy. We relate equilibrium returns on bonds of different maturities under the optimal policy to criteria commonly used by public debt management offices. In particular, our results suggest that the government should not simply focus on minimizing expected funding costs by tilting issuance towards maturities with the lowest expected returns. Interestingly however, a policy that responds to shifts in the yield curve may be consistent with optimal policy in our framework.

**Related Literature.** The role of clienteles was emphasized early on in the term-structure literature. The preferred-habitat hypothesis of Culbertson (1957) and Modigliani and Sutch (1966) posits that there are clienteles with preferences for specific maturities and that these clienteles

\(^3\)Note that even if the government does not issue two-period bonds, we allow these to be traded in zero net supply. Thus, the government does not change the set of tradable securities but it does affect the net supply through its choice of maturity structure.

\(^4\)Throughout we use the criterion of *ex ante* efficiency as opposed to *interim* efficiency, identifying generations by the period in which they are born as opposed to the period and the state in which they are born.

\(^5\)This assumes that clienteles are modeled through standard preferences, i.e., ignoring constraints or other institutional frictions.
can engage in only limited substitution across maturities. More recently, Vayanos and Vila (2009) develop a formal model of preferred habitat in which each maturity has its own clientele, and substitution across maturities is carried out by risk-averse arbitrageurs. They assume that clienteles are infinitely risk-averse over consumption at their desired maturity. We instead model clienteles through CRRA preferences, dispense with arbitrageurs, and perform a normative analysis of maturity structure.

Recent empirical studies document supply effects which are consistent with the existence of clienteles for government bonds in general and for government bonds of different maturities in particular. Greenwood and Vayanos (2008) show that the slope of the yield curve is positively related to the average maturity of government debt, consistent with our theoretical results. Krishnamurthy and Vissing-Jorgensen (2009) show that the corporate spread is negatively related to the supply of government debt, i.e., government debt is expensive relative to corporate when it is in short supply. Greenwood, Hanson and Stein (2010) provide evidence that the corporate sector responds to changes in the maturity structure of government debt by issuing more at the short end when the government issues more long-term debt and vice-versa.

Our normative analysis relates to the literature on optimal public debt policy. The benchmark in this literature is the Ricardian equivalence result of Barro (1974): in a representative-agent model with non-distortionary taxes, the level and composition of government debt are irrelevant. To get around the irrelevance result, the literature has emphasized the distortionary aspect of taxes. With distortionary taxes, the government can improve welfare by using debt to smooth taxes over time and across states of the world. Barro (1979) and Aiyagari, Marcet, Sargent and Seppälä (2002) show that distortionary taxes imply an optimal time path for the level of government debt in a framework with one-period non-contingent bonds. In the presence of multiple securities, distortionary taxes also imply an optimal composition of the government debt portfolio. Lucas and Stokey (1983) derive the optimal portfolio in terms of Arrow-Debreu securities. Angeletos (2002) and Buera and Nicolini (2004) show how this first-best outcome can be implemented with non-contingent bonds of different maturities, provided that there are bonds of as many maturities as states of nature so that markets are complete. Nosbusch (2008) derives the optimal maturity structure under incomplete markets. Faraglia, Marcet and Scott (2008) extend the complete markets analysis of optimal maturity structure to a framework with capital accumulation. The general idea in these papers is that the optimal debt portfolio is chosen so that its value is negatively correlated with shocks to government spending. This literature assumes a representative agent, thus abstracting

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6 This positive relationship would also arise in a two-period OLG model.

7 Newman and Rierson (2004) present evidence from the European telecommunication-sector bond market showing that a new bond issue by one firm can temporarily increase yield spreads on other bonds in the sector. This is again consistent with the existence of clienteles in bond markets.

8 A separate strand of the literature considers optimal debt policy when debt contracts are nominal and the government has control over inflation. In this case, the government can use state contingent inflation to achieve
from clienteles. In contrast, we assume heterogenous agents and non-distortionary taxes, and we characterize the optimal maturity structure in terms of intergenerational risksharing.

It is well known that Ricardian equivalence fails in models with overlapping generations, along the lines of Samuelson (1958), Diamond (1965) and Blanchard (1985). With overlapping generations, the timing of debt and taxes matters because debt shifts the tax burden to future generations.\textsuperscript{9,10} Fischer (1983) and Gale (1990) show how debt can be used for risksharing across generations in a stochastic 2-period OLG model. In particular, Gale (1990) shows that by introducing new securities in the form of long-dated noncontingent bonds the government can improve intergenerational risksharing under certain conditions.\textsuperscript{11} Our analysis differs because we allow for heterogeneous investment horizons and clienteles, we assume that the government does not change the set of traded securities but merely their net supply, and we give an explicit characterization of the optimal maturity structure.

An alternative way to achieve intergenerational risksharing is through the social security system. Ball and Mankiw (2007) show how social security can be used to implement risksharing contracts that all generations would be willing to sign if they were able to meet behind a Rawlsian “veil of ignorance.”\textsuperscript{12} Our setup differs in two key respects. First, as already pointed out in the previous paragraph, our assumption that generations live for three rather than two periods leads naturally to the existence of clienteles with different investment horizons. Second, we focus on using the maturity composition of government debt to achieve efficient intergenerational risksharing. An advantage of this approach is that it is politically easier to implement than a system which relies purely on social security. In particular, a pure social security system would need to cut benefits to current voters in period $j$ in some states of the world \textit{ex post}, which is politically difficult. More specifically, Rangel and Zeckhauser (2001) model the electoral process in an OLG economy with social security and show that there are multiple equilibria in this type of setting. In particular, there is always an equilibrium where different generations do not engage in risksharing through social security and instead use their voting power to expropriate each other. In contrast, in the implementation with government debt which we propose, generations $j-2$ and $j-1$ would have voluntarily chosen a bond portfolio \textit{before} the realization of uncertainty which results in a capital loss in the required states of the world.

tax smoothing. Prominent examples of this approach include Lucas and Stokey (1983), Bohn (1988), Calvo and Guidotti (1990,1992), Barro (2002), Benigno and Woodford (2003), and Lustig, Sleet and Yeltekin (2008). Missale and Blanchard (1994) show that with nominal debt contracts, the optimal maturity can be decreasing in the total size of the debt.\textsuperscript{9} This holds even when generations are infinitely lived, as shown by Buiter (1988) and Weil (1989).\textsuperscript{10} In the presence of capital, debt also matters because it affects capital accumulation.\textsuperscript{11} Weiss (1980) and Bhattacharya (1982) show that in a stochastic OLG framework with money, state-contingent inflation can be used to share risks across generations.\textsuperscript{12} Campbell and Nosbusch (2007) study how a social security system that shares risks across generations affects equilibrium asset prices.
2 Model

2.1 Private Agents

We consider an overlapping generations model where agents live for three periods. Each period $t$, a new generation is born and we index generations by the period in which they are born. Hence generation $t$ refers to the generation which is young in period $t$, middle-aged in period $t+1$, and old in period $t+2$. Each generation receives an endowment $\alpha_t$ when young and consumes $c_{t+2}$ when old in period $t+2$. We assume that utility over final consumption is CRRA with the same coefficient of relative risk aversion $\gamma$ for all generations:

$$u(c) \equiv \frac{c^{1-\gamma}}{1-\gamma}.$$

In a given period $t$, three generations are alive: the young (generation $t$), the middle-aged (generation $t-1$), and the old (generation $t-2$). The old just consume their accumulated wealth. The young and the middle-aged invest in order to finance consumption in old age. They can invest in a linear technology and trade in non-contingent bonds of different maturities.

The gross one-period rate of return on the linear technology is constant and equal to $R > 1$ in all periods except in period $j$ (throughout the paper we use $t$ to denote a generic time period and $j$ to denote the period of the shock). We assume that there are two possible states denoted by $s = H, L$ with respective probabilities $p$ and $1-p$. In state $H$, the gross return between periods $j$ and $j+1$, denoted by $R_s$, takes a high value $R_H$ while in state $L$, $R_s$ takes a low value $R_L$. This is the only source of uncertainty in the model. The linear technology is riskless in period $j$ in the sense that the uncertainty about $R_s$ is resolved in period $j$. Figure 1 summarizes endowments and consumption of the different generations around the period of the shock.

Generations born before period $j-2$ do not live long enough to see the realization of uncertainty and there is no motive for them to trade in our framework. We assume that in period $j-1$, before the realization of the shock, generations $j-2$ and $j-1$, who are alive at that time, can trade one-period and two-period bonds. Since there are only two states of nature, allowing for two maturities completes the market from the perspective of period $j-1$. From period $j$ onwards, generations present in the market only trade one-period bonds. This is without loss of generality since long-maturity bonds are redundant after uncertainty is resolved in period $j$. 


<table>
<thead>
<tr>
<th>Period</th>
<th>$j - 2$</th>
<th>$j - 1$</th>
<th>$j$</th>
<th>$j + 1$</th>
<th>$j + 2$</th>
</tr>
</thead>
</table>

**Figure 1: Life-cycles and return shock.**

### 2.2 Government

The government affects the net supply of one-period and two-period bonds through its issuance policy. Since it is impossible to share risks with generations who die before the realization of uncertainty, we assume that the government does not intervene in the bond market before period $j - 1$. In period $j - 1$, before the realization of the shock, the government can trade one-period and two-period bonds with generations $j - 1$ and $j - 2$ who are the investors present in the market at that time. From period $j$ onwards, the government only trades one-period bonds with generations present in the market in those periods. We denote by $b_t$ the face value of one-period debt issued in period $t - 1$ and maturing in period $t$, for $t \geq j$. We denote by $B$ the face value of two-period debt issued before the shock in period $j - 1$ and maturing in period $j + 1$.

For simplicity we set government spending equal to zero. Assuming positive levels of government spending would not change the intuition of the results. Hence, the government budget constraint in period $j - 1$ is

$$\frac{b_j}{R} + \frac{B}{L^2} = 0,$$  \hspace{1cm} (1)

where $L$ is the yield on two-period bonds, which is determined in equilibrium. For a given choice of maturity structure $(b_j, B)$ in period $j - 1$, the value of the government debt position in period $j$
and state $s$ is

$$D_{j,s} \equiv b_j + \frac{B_s}{R_s}, \quad s = H, L. \quad (2)$$

We assume that the government can levy lump-sum taxes $\delta_{t,s}$ on the endowments of young generations from period $j$ onwards. A negative value ($\delta_{t,s} < 0$) is interpreted as a lump-sum transfer to the young generation in period $t$. The evolution of the face value of one-period debt after the shock is thus given by

$$b_{j+1,s} = R_s (D_{j,s} - \delta_{j,s}), \quad (3)$$

$$b_{t+1,s} = R (b_{t,s} - \delta_{t,s}), \quad \text{for all } t \geq j + 1. \quad (4)$$

3 Demand and Supply Effects in the Government Bond Market

In this section, we characterize equilibrium in the government bond market in the period before the realization of the shock. We derive the equilibrium yield curve and analyze how it depends on the demand by different generations, i.e., the clientele mix, and on the supply of two-period bonds.

3.1 Debt Holdings and Equilibrium

In period $j - 1$, only two generations are present to trade bonds with the government: the middle-aged generation $j - 2$ and the young generation $j - 1$. Let $B^{j-2}$ and $B^{j-1}$ denote the face values of their holdings of two-period bonds. For given holdings $B^{j-2}$, generation $j - 2$’s consumption is given by

$$c_{j,s} = R^2 \alpha_{j-2} + \left( \frac{L^2}{R_s} - R \right) \frac{B^{j-2}}{L^2}. \quad (5)$$

The first term in (5) corresponds to consumption of generation $j - 2$ under autarky. The second term captures the impact of long-maturity bond holdings by generation $j - 2$ on its consumption in period $j$. It is equal to the one-period excess return of two-period bonds over one-period bonds, multiplied by the market value of two-period bonds acquired by generation $j - 2$. Generation $j - 2$ chooses its holdings of two-period bonds to maximize $E[u(c_j)]$ subject to (5). Hence $B^{j-2}$ is
determined by the first-order condition

$$
E \left\{ u' \left[ R^2 \alpha_{j-2} + \left( \frac{L^2}{R_s} - R \right) \frac{B^{j-2}}{L^2} \right] \left( \frac{1}{R_s} - \frac{R}{L^2} \right) \right\} = 0. \tag{6}
$$

Eq. (6) links generation $j - 2$’s demand for two-period bonds to the two-period interest rate. Since $u'(0) = \infty$, (6) has a solution $B^{j-2}$ for any value of $L$ satisfying no-arbitrage, i.e., for any $L \in (RR_L, RR_H)$. Moreover, the solution is unique since utility is strictly concave.

Similarly, for given holdings $B^{j-1}$, generation $j - 1$’s consumption is

$$
c_{j+1,s} = RR_s \alpha_{j-1} + \left( L^2 - RR_s \right) \frac{B^{j-1}}{L^2}. \tag{7}
$$

The first term in (7) corresponds to consumption of generation $j - 1$ under autarky and the second term captures the impact on consumption in period $j + 1$ of trading in two-period bonds in period $j - 1$. It is the difference in returns between a strategy of buying a two-period bond and holding it until maturity and a strategy of investing at the one-period riskfree rate and rolling over in period $j$, multiplied by the initial market value of two-period bonds held by generation $j - 1$. Hence, the optimal demand for two-period bonds by the young generation in period $j - 1$ is determined by

$$
E \left\{ u' \left[ RR_s \alpha_{j-1} + \left( L^2 - RR_s \right) \frac{B^{j-1}}{L^2} \right] \left( 1 - \frac{RR_s}{L^2} \right) \right\} = 0. \tag{8}
$$

Again there is a unique solution to this first-order condition. In equilibrium the yield on two-period bonds needs to adjust to clear the market: $B^{j-2}(L) + B^{j-1}(L) = B$. In the Appendix we prove the following proposition.

**Proposition 1.** There exists a unique equilibrium in the government bond market in period $j - 1$.

The argument relies on the fact that demands for two-period bonds are strictly increasing in $L$, converge to $-\infty$ when $L^2$ goes to $RR_L$, and converge to $\infty$ when $L^2$ goes to $RR_H$. This implies that there exists a unique value of $L$ that equates the demand for two-period bonds to their supply.

### 3.2 Effect of Clienteles on the Yield Curve

Clienteles arise naturally in our setup because different generations alive in any given period vary in their investment horizon. In particular, in period $j - 1$, the young generation $j - 1$ can be thought of as a long-horizon clientele whereas the middle-aged generation $j - 2$ can be thought of
as a short-horizon clientele. In this section, we analyze the impact of the clientele mix on the yield curve by changing the relative size of the endowments of the two clienteles, holding the value of their aggregate endowment (measured in period $j-1$) constant.

**Proposition 2.** If $\gamma > 1$, then an increase in the relative wealth of the long-horizon clientele in period $j-1$ (i.e., an increase in $\alpha_{j-1}$, holding $R\alpha_{j-2}+\alpha_{j-1}$ constant) lowers the two-period interest rate. The result is reversed if $\gamma < 1$.

According to the preferred-habitat hypothesis originally formulated by Culbertson (1957) and Modigliani and Sutch (1966), short-term bonds are demanded mainly by short-horizon investors, while long-term bonds are demanded by long-horizon investors. Therefore, when generation $j-1$ commands more resources, two-period bonds should be in higher demand and thus more expensive. Proposition 2 confirms this intuition when agents’ coefficient of relative risk aversion $\gamma$ is strictly larger than one. However the effect is reversed when $\gamma < 1$, and when $\gamma = 1$ (logarithmic utility) the clientele mix has no effect on the equilibrium yield curve. The reason is that $\gamma$ needs to be strictly greater than unity for generation $j-1$ to invest a higher share of its endowment in two-period bonds than generation $j-2$. In particular, when utility is logarithmic, agents behave myopically and their portfolio choice is independent of their investment horizon. Thus in our framework, we find that CRRA utility with $\gamma \leq 1$ cannot generate intuitive clientele effects. Since such effects seem important in practice, our results support the assumption $\gamma > 1$. This is consistent with the literature on the equity premium puzzle but interestingly it is at odds with the assumption of logarithmic utility commonly used in term-structure models.

### 3.3 Effect of Supply on the Yield Curve

Since demand for long-term bonds is increasing in $L$, an increase in the supply of long-term bonds raises the yield of two-period bonds and the slope of the yield curve.

**Proposition 3.** An increase in the supply $B$ of two-period bonds raises the equilibrium level of $L$.

The intuition for this result is the following. Any incremental issuance of long-term debt has to be absorbed by generations $j-2$ and $j-1$ since they are the only investors present in the market in period $j-1$. Lengthening the maturity structure shifts their consumption towards the state where $R_s$ is low. This lowers their valuation of two-period bonds. The only way the market can clear is for $L$ to rise. It is crucial for the supply effect to obtain that generations $j-2$ and $j-1$ do not experience offsetting changes in future tax rates, so that their consumption allocation is sensitive to changes in the maturity structure of debt. This effect cannot arise in a representative agent model.\(^{13}\)

\(^{13}\)A representative agent would anticipate that future taxes would cancel out any gains and losses on the agent’s
Since an increase in supply raises $L$, it also raises the expected excess return of two-period bonds relative to one-period bonds. We show in Section 5 that under the optimal supply of two-period bonds, this excess return is negative over a two-period horizon, and possibly over a one-period horizon as well. In general, the long-term bond premium can be positive or negative, depending on $B$. If two-period bonds are in large supply, they trade at a low price in period $j - 1$, and their expected return exceeds that of one-period bonds over a one- and a two-period horizon. Conversely, if supply is small, then two-period bonds return less on average than one-period bonds.

## 4 A Clientele-Based Theory of the Optimal Maturity Structure

In general, intergenerational risksharing is inefficient in an OLG economy. This is also the case in our setup. More specifically, in the absence of government the only risksharing that can occur is between generations $j - 2$ and $j - 1$. These two groups can enter a transaction in period $j - 1$ before the realization of the shock, whereby generation $j - 1$ transfers some of the rate of return risk to generation $j - 2$. This can be achieved for instance by generation $j - 2$ selling a long-term bond to generation $j - 1$ or it could take the form of an interest rate derivative. In either case, this does not require government intervention since these securities would be in zero net supply. This private risksharing arrangement would guarantee that

\[
\frac{u'(c_{j,H})}{u'(c_{j,L})} = \frac{R_H}{R_L} \frac{u'(c_{j+1,H})}{u'(c_{j+1,L})}.
\]

However, the resulting allocation is still suboptimal. First, it leaves generation $j$ fully exposed to the shock. Moreover it does not give any exposure to the shock to generations born after period $j$. Further risksharing in the absence of government intervention is impaired by a fundamental form of market incompleteness: one cannot contract ex ante with generations born after the realization of the shock.

In this section, we derive the complete markets allocation that would arise if all generations were able to trade in one- and two-period bonds in period $j - 1$ (Section 4.1). Then we show how the government can use the maturity structure to implement this allocation (Section 4.2). The observation that the government can replicate the complete markets allocation plays a crucial role in our analysis.
4.1 Complete Markets Equilibrium

In this subsection, we consider a hypothetical benchmark case where there is no government and all generations born from period \( j - 2 \) onwards participate in the market in period \( j - 1 \), trading one- and two-period bonds (in zero net supply) amongst each other. In this case the resulting equilibrium allocation is ex ante efficient. Indeed, the combination of the riskless technology together with one- and two-period bonds spans the state space and therefore allows to reach an efficient allocation, provided that all generations are present for trading before the realization of the shock.

Let \( B^t \) denote the face value of two-period bonds held by generation \( t \geq j - 2 \). The consumption of generations \( j - 2 \) and \( j - 1 \) is given by (5) and (7). The consumption of generation \( j \) is

\[
c_{j+2,s} = RR_s \alpha_j + R \left( L^2 - RR_s \right) \frac{B^j}{L^2},
\]

(10)

while the consumption of generation \( t \geq j + 1 \) is

\[
c_{t+2,s} = R^2 \alpha_t + R^{t+2-(j+1)} \left( L^2 - RR_s \right) \frac{B^t}{L^2}, \quad t \geq j + 1.
\]

(11)

The interpretation of (10) and (11) is analogous to that of Eq. (7). The first term corresponds to consumption under autarky and the second term captures the impact of trading in two-period bonds. The latter depends on the difference in returns between a strategy of buying a two-period bond in period \( j - 1 \) and holding it until maturity and a strategy of investing at the one-period riskfree rate and rolling over in period \( j \).

**Definition 1.** A complete markets equilibrium consists of consumption allocations \( \{c^*_t,s\}_{t=j-2}^{\infty} \), two-period bond holdings \( \{\tilde{B}^t\}_{t=j-2}^{\infty} \) and a two-period interest rate \( L^* \) such that all generations choose their holdings of two-period bonds to maximize their expected utility of consumption and the market for two-period bonds clears

\[
\sum_{t=j-2}^{\infty} \tilde{B}^t = 0.
\]

(12)

We prove in the Appendix that there is a unique complete markets equilibrium. Furthermore, since the complete markets equilibrium is ex ante efficient, there exists a set of weights \( \{\lambda_t\}_{t=0}^{\infty} \) such that the complete markets equilibrium consumption allocation maximizes the weighted sum
of expected utilities

$$\mathbb{E} \sum_{i=0}^{\infty} \lambda_i u(c_{j+i,s})$$  \hspace{1cm} (13)$$

subject to state-dependent intertemporal budget constraints

$$c_{j,s} + \frac{1}{R_s} \sum_{i=0}^{\infty} \frac{c_{j+1+i,s}}{R^i} = \alpha_{j-2} R^2 + \alpha_{j-1} R + \alpha_j + \frac{1}{R_s} \sum_{i=0}^{\infty} \alpha_{j+1+i} R^{-i} \equiv e_{j,s}, \text{ } s = H, L. \hspace{1cm} (14)$$

Let $P$ denote this planning problem. Eq. (14) can be derived by adding the budget constraints of all generations $t \geq j - 2$. We assume that the production technology cannot be run in reverse, which implies non-negativity constraints on aggregate investment in the technology. We proceed under the assumption that these constraints are not binding (this is indeed the case in our benchmark calibration).\footnote{More precisely, define $W_j = \alpha_{j-2} R^2 + \alpha_{j-1} R + \alpha_j$, $W_{j+1,s} = R_s(W_j - c_{j,s}) + \alpha_{j+1}$, and for $t \geq j + 1$, $W_{t+1,s} = R(W_{t,s} - c_{t,s}) + \alpha_{t+1}$. Non-negativity constraints on aggregate investment require that $c_t \leq W_t$ for all $t \geq j$.} The solution to $P$ determines the efficient consumption allocation as a function of relative weights. The weights supporting the complete markets equilibrium in turn depend on endowments and can be determined through budget constraints.

**Lemma 1.** Given a set of weights in $P$, the consumption of generation $j - 2$ is given by

$$c_{j,s}^* = \frac{1}{1 + \beta_j^{-1} R_s - 1} e_{j,s},$$  \hspace{1cm} (15)$$

where $\beta_j = \left[ \sum_{i=1}^{\infty} \left( \frac{1}{R_s-1} \right)^{1-\frac{1}{\gamma}} \left( \frac{\lambda_0}{\lambda_i} \right)^{-\frac{1}{\gamma}} \right]^{\gamma}$. Similarly the consumption of generation $j - 1$ is given by

$$c_{j+1,s}^* = \frac{1}{1 + \beta_{j+1}^{-1} R_s - 1} R_s(e_{j,s} - c_{j,s}^*),$$  \hspace{1cm} (16)$$

where $\beta_{j+1} = \left[ \sum_{i=1}^{\infty} \left( \frac{1}{R_s-1} \right)^{1-\frac{1}{\gamma}} \left( \frac{\lambda_1}{\lambda_i+1} \right)^{-\frac{1}{\gamma}} \right]^{\gamma}$. Finally, for any $k \geq 2$

$$\frac{c_{j+k,s}^*}{c_{j+k+1,s}^*} = \left( \frac{\lambda_{k-1}}{\lambda_k R} \right)^{-\frac{1}{\gamma}}, \text{ } s = H, L. \hspace{1cm} (17)$$
Lemma 1 indicates how the efficient allocation can be obtained recursively given a set of Pareto weights. Eq. (15) can be given an intuitive interpretation. Indeed, the planner’s objective in period \( j \) and state \( s \) can be written as a two-period consumption saving problem

\[
\max_{c_{j,s}} u(c_{j,s}) + \beta_j u \left( \sum_{i=1}^{\infty} \frac{c_{j+i,s}}{R^{i-1}} \right)
\]

subject to \( \sum_{i=1}^{\infty} (c_{j+i,s}/R^{i-1}) = R_s (c_{j,s} - c_{j,s}) \). The factor \( \beta_j \) thus captures the weight of all generations born in period \( j - 1 \) and later relative to generation \( j - 2 \). In a similar fashion, the factor \( \beta_{j+1} \) in (16) captures the weight of all generations born in period \( j \) and later relative to generation \( j - 1 \). From period \( j + 1 \) onwards, ex ante efficiency requires that the ratio of consumption allocations over two consecutive periods be constant across states.

**Lemma 2.** The relative weights \( \beta_j \) and \( \beta_{j+1} \) supporting the complete markets allocation solve the following two non-linear equations

\[
\mathbb{E}[u'(c^*_j,s)c^*_j] = R^2 \alpha_{j-2} \mathbb{E}[u'(c^*_j,s)],
\]

\[
\mathbb{E}[u'(c^*_{j+1,s})c^*_{j+1,s}] = R \frac{\mathbb{E}[u'(c^*_j,s)]}{\mathbb{E}[\frac{1}{R_s} u'(c^*_j,s)]} \alpha_{j-1} \mathbb{E}[u'(c^*_j+1,s)].
\]

Furthermore,

\[
\left( \frac{\lambda_{k-1}}{\lambda_k R} \right)^{-\frac{1}{\gamma}} = \frac{\alpha_{j+k-2}}{\alpha_{j+k-3}}, \quad \text{for } k = 2 \text{ or } k \geq 4,
\]

\[
= R \frac{\mathbb{E}[\frac{1}{R_s} u'(c^*_j,s)]}{\mathbb{E}[u'(c^*_j,s)]} \alpha_{j+1} \alpha_{j}, \quad \text{for } k = 3.
\]

Together with Eqs. (15), (16) and (17), Lemma 2 shows how to construct the complete markets consumption allocation recursively. Eq. (19), derived from the budget constraint and first-order condition of generation \( j - 2 \), determines \( \beta_j \) and \( c^*_j,s \). The relative weights that support the complete markets equilibrium essentially depend on the relative endowments of generations, adjusting for differences in the rates of return they are exposed to under autarky where appropriate. The two-
period interest rate under the complete markets equilibrium follows from (6)

\[(L^*)^2 = \frac{\mathbb{E}[u'(c^*_{j,s})]}{\mathbb{E}\left[\frac{1}{R_s} u'(c^*_{j,s})\right]} R.\]  \hspace{1cm} (23)

We discuss the excess returns earned by two-period bonds over one- and two-period horizons under the complete markets allocation in Section 5.2. The analysis of risk premia is related to our next proposition, which characterizes how the consumption of all generations \(t \geq j - 2\) is affected by the interest rate shock realization in the complete markets equilibrium.

**Proposition 4.** Under complete markets, generations born in period \(j - 1\) and later consume more in state \(H\) than in state \(L\). Generation \(j - 2\) consumes more in state \(H\) than in state \(L\) if \(\gamma\) is sufficiently large, but the comparison is reversed if \(\gamma \leq 1\).

The dependence of consumption on the interest rate shock realization is a key qualitative feature of the complete markets allocation. It can be understood from the social planner’s problem \(\mathcal{P}\). Since a high interest rate in period \(j\) raises aggregate resources, consumption is unambiguously higher in periods \(t \geq j + 1\) once the interest rate has returned to its normal level \(R\). The effect on consumption by generation \(j - 2\) in period \(j\) is ambiguous. High interest rates induce the planner to substitute consumption towards later periods, and this tends to reduce consumption in period \(j\). At the same time, the income effect induces the planner to raise consumption in all periods, including \(j\). Finally, the income effect is tempered by a wealth effect arising because high interest rates reduce the present value of endowments evaluated as of period \(j\) (indeed, \(e_{j,H} < e_{j,L}\)). When \(\gamma\) is sufficiently large, the income effect dominates, and therefore generation \(j - 2\) consumes more when interest rates are high. When instead \(\gamma \leq 1\), the substitution and wealth effects dominate.\(^{15,16}\)

Alternatively, Proposition 4 can be interpreted in terms of the holdings \(\{\tilde{B}^t\}_{t=j-2}^{\infty}\) of two-period bonds under the decentralized complete markets equilibrium. Proposition 4 implies that generations born in period \(j + 1\) and later are net sellers of two-period bonds, thus providing insurance to generations \(j - 1\) and \(j\) who are naturally exposed to the rate of return shock and are therefore willing to take long positions in two-period bonds to partially hedge their risks. The proposition also implies that generation \(j - 2\) is a net seller of two-period bonds for \(\gamma\) large enough and a net buyer if \(\gamma \leq 1\). The reason why generations that do not bear risk under autarky are willing to take exposure to interest rate risk is that they are compensated by higher expected consumption. When \(\gamma\) is large, generations \(j - 1\) and \(j\) have a strong motive to hedge against low interest rates. This drives the price of two-period bonds up and their expected excess return is

\(^{15}\)In our benchmark calibration, the threshold value of \(\gamma\) above which the income effect dominates is close to 3.

\(^{16}\)The marginal impact of \(R_s\) on \(c_{j,s}\) can be obtained by differentiation of (15).
negative both over one and two periods, and therefore all generations that do not bear risk under autarky provide partial insurance to generations $j-1$ and $j$. When $\gamma \leq 1$, generations $j-1$ and $j$ are less eager to hedge, the price of two-period bonds is lower and the expected excess return of two-period bonds goes up. In this case, generations born in period $j+1$ and later sell two-period bonds whereas generation $j-2$ is a net buyer. This can happen in equilibrium because as discussed in Section 5.2, expected excess returns on two-period bonds are negative over two periods while being positive over one period because of a convexity effect.

Figure 2 depicts the state-contingent consumption of generations born in period $j-2$ and later, normalized by $R^2$, both under autarky and in the complete markets equilibrium. For the purpose of constructing this figure, we assume that endowments are constant, $\alpha_t = 1$ for all $t$, the two states are equally likely, $p = .5$, and $\gamma = 4$. We interpret one period to last for 15 years and set the average real interest rate equal to 2.1% per year, the historical average for the United States in the twentieth century$^{17}$. We assume that in state $H$ the rate of return on the linear technology is 10% above its mean, and in state $L$ it is 10% below its mean. Under autarky only generations $j-1$ and $j$ are exposed to the shock. Under complete markets by contrast, all generations born in period $j-2$ and later bear some risk. In our calibration for $\gamma = 4$, generation $j-2$ as well as all generations $t \geq j+1$ provide insurance by selling two-period bonds and are compensated by higher expected consumption.

![Figure 2: Complete markets consumption allocation.](image)

$^{17}$Source: Dimson, Marsh, and Staunton (2002).
4.2 Implementing Efficient Risksharing with Government Debt

We now return to the more realistic case where only generations $j-2$ and $j-1$ trade in the market in the period before the shock. The fundamental incompleteness problem caused by the non-participation of unborn generations leaves scope for government intervention. In this subsection, we demonstrate that the maturity structure of government debt can be used to make state-contingent intergenerational transfers and implement the complete markets allocation.

**Proposition 5.** There exists an amount of two-period debt $B^*$ that allows the government to implement efficient intergenerational risksharing.

Proposition 5 can be proved by construction. Suppose the government sets $B$ equal to

\[ B^* \equiv \tilde{B}^{j-2} + \tilde{B}^{j-1}, \quad (24) \]

i.e., the quantity of two-period bonds that generations $j-2$ and $j-1$ buy in the complete markets equilibrium. Then the two-period interest rate $L$ is as in that equilibrium, and so are the consumption allocations of generations $j-2$ and $j-1$. The government’s corresponding position in one-period debt $b^* = -[R/(L^*)^2]B^*$ follows from the government’s budget constraint (1). Furthermore, taxes and transfers $\delta^*_{t,s}$ for $t \geq j$ are determined by

\[ c^*_{j+2,s} = RR_s(\alpha_j - \delta^*_{j,s}), \quad (25) \]
\[ c^*_{t+2,s} = R^2(\alpha_t - \delta^*_{t,s}), \quad \text{for } t \geq j + 1. \quad (26) \]

The evolution of short term debt after period $j$ is pinned down by Eqs. (3) and (4).

To gain more intuition about $B^*$, we use the market clearing condition (12) to write

\[ B^* = -\sum_{t=j}^{\infty} \tilde{B}^t. \quad (27) \]

The interpretation of Eq. (27) is that in order to implement the complete markets allocation, the government replicates the trades that generations born from period $j$ onwards would engage in if they were present in the market in period $j-1$. Put differently, the government can effectively trade on behalf of unborn generations through its choice of the net supply of two-period bonds.

It is worth noting that the government intervention in bond markets merely consists in supplying the right quantity of long-term debt rather than changing the set of traded securities. Naturally
in order to engineer the right state contingent capital gains and losses on the debt portfolios held by generations $j-2$ and $j-1$, it is essential that long-maturity bonds are available. In a setting with only short-term debt, the value of the bonds held by generations $j-2$ and $j-1$ would be independent of the state in period $j$. Proposition 5 is related to the results of Angeletos (2002) and Buera and Nicolini (2004). Their analysis of optimal maturity structure relies on the idea of using noncontingent long-maturity bonds to implement an efficient allocation in a representative agent framework with distortionary taxes. In their context the government debt maturity structure is used to smooth taxes optimally, whereas in our OLG setup with lump-sum taxes and transfers the maturity structure is used to achieve efficient intergenerational risksharing.

We now return to the calibration used for Figure 2 in Section 4.1 and illustrate how the government implements the efficient allocation each period while satisfying its budget constraint. In the complete markets equilibrium, for $\gamma = 4$ generation $j-2$ is a seller of two-period bonds, $\tilde{B}_{j-2} < 0$. Generation $j-1$ on the other hand hedges some of its existing exposure by buying an amount of two-period bonds $\tilde{B}_{j-1} > 0$. The net holdings of two-period bonds by these two generations is positive in our calibration, i.e., generations $j$ onwards are net sellers of two-period bonds. As a result, the government should be a net issuer of two-period bonds in period $j-1$ under incomplete markets, $B^* = .85$. In period $j$ the shock realizes. Figure 3 depicts government debt and tax policies in each of the two states from period $j$ onwards. The value of the government debt position in period $j$ is described by Eq. (2). If the economy is in state $H$, the government (having issued two-period bonds) makes a capital gain on its bond portfolio. Figure 3 illustrates that in this case, the government taxes generation $j$ ($\delta_{j,H} > 0$) and invests its entire portfolio in short-term bonds. From then on, the government just rolls over a constant amount of short-term assets: $b_{t,H} = b^H < 0$, for all $t \geq j+1$. Each period it pays out the interest as a transfer to the young generation $\delta_{t,H} = \delta^H < 0$, for all $t \geq j+1$. Conversely, if the economy is in state $L$, the government makes a capital loss on its bond portfolio. In this case, the government makes a transfer to generation $j$ ($\delta_{j,L} < 0$) and finances this by adding to its stock of short-term debt. From then on, the government just rolls over a constant amount of short-term debt: $b_{t,L} = b^L > 0$, for all $t \geq j+1$. Each period it pays the interest by imposing a tax on the young generation $\delta_{t,L} = \delta^L > 0$, for all $t \geq j+1$. Thus generation $j$’s exposure to the shock is reduced while all generations $t \geq j+1$ get some exposure to the shock.

4.3 Alternative Risksharing Mechanisms

In principle, the government can implement efficient intergenerational risksharing through other arrangements which do not require using the maturity structure of government debt. A natural
Figure 3: One-period debt issuance and transfers.

alternative would be social security. For example, the government could implement the efficient outcome by imposing a flat tax on the endowment of all young generations starting with generation \( j - 2 \) and paying a state contingent social security benefit to old generations.

A key advantage of using the maturity structure of government debt to achieve the efficient outcome is that it is politically easier to implement than a system that relies purely on social security. In particular, a social security system would need to cut social security benefits of generations \( j - 2 \) and \( j - 1 \) in some states of the world in period \( j \). This is likely to be politically difficult as these are current voters. Rangel and Zeckhauser (2001) model the electoral process in a two-period OLG economy assuming that each generation maximizes their own welfare and assume and that all agents alive in any given period vote on the amount of intergenerational redistribution. They show that if the median voter is old, there is no intergenerational risksharing since the old are unaffected by future policy and thus it is optimal for them to expropriate the young to the extent that they can. If the median voter is young, there exists an equilibrium in which the young are willing to implement the optimal transfers as long as they expect future generations to do the same. However this requires the threat of expropriation if they deviate from the equilibrium strategy. More importantly, there is always another equilibrium where different generations do not engage in any risksharing through social security and instead use their voting power to expropriate each other. The authors conclude that this type of institution cannot be relied upon to achieve
efficient intergenerational risksharing. In contrast, in the implementation with government bonds of different maturities that we propose, generations \( j - 2 \) and \( j - 1 \) would have voluntarily chosen a bond portfolio before the realization of uncertainty which results in a capital loss in the required states of the world.

5 Properties of the Optimal Debt Policy

5.1 Optimal Government Supply of Two-Period Bonds

In this section, we characterize further the optimal supply of two-period bonds \( B^* \) by the government and how it depends on the relative wealth of generations \( j - 2 \) and \( j - 1 \). First, we analyze the sign of the optimal supply of two-period debt by the government.

**Proposition 6.** The optimal supply of two-period bonds is positive if \( \gamma \leq 1 \). If \( \gamma \) is large, then the optimal supply \( B^* \) of two-period bonds is positive when the relative wealth of the long-horizon clientele is large.

This can be understood from Eq. (24). Generation \( j - 1 \) is exposed to the shock and it wants to hedge partially against this risk by buying two-period bonds, i.e., \( \tilde{B}^{j-1} > 0 \). Proposition 4 shows that for \( \gamma \leq 1 \), \( c_{j,H}^* < c_{j,L}^* \). Since the consumption of generation \( j - 2 \) is riskless under autarky, generation \( j - 2 \) also buys two-period bonds to get its optimal exposure to the shock in this case, i.e., \( \tilde{B}^{j-2} > 0 \). Hence, the optimal supply of two-period bonds by the government \( B^* \) is unambiguously positive for \( \gamma \leq 1 \). On the other hand, when \( \gamma \) is large, Proposition 4 shows that \( c_{j,H}^* > c_{j,L}^* \). This implies that generation \( j - 2 \) needs to issue two-period bonds in order to get its optimal exposure to the shock, i.e., \( \tilde{B}^{j-2} < 0 \). In this case, the optimal supply of bonds by the government, \( B^* \), can be positive or negative depending on the relative wealth of generations \( j - 2 \) and \( j - 1 \). Alternatively, the result can be understood from Eq. (27). The government effectively trades two-period bonds on behalf of the generations born from period \( j \) onwards. Proposition 4 shows that for all \( t \geq j + 1 \), \( c_t,H > c_t,L \). This implies that for \( t \geq j + 1 \), \( \tilde{B}^t < 0 \), i.e., generations born at date \( j + 1 \) or later would be willing to provide insurance by issuing two-period bonds. The sign of \( B^* \) is ambiguous because generation \( j \) would buy two-period bonds to hedge some of its exposure, i.e., \( \tilde{B}^j > 0 \).

Our model also provides a clear characterization of the impact of changes in the clientele mix on the optimal maturity composition. This can be obtained by varying the relative size of the endowments of generations \( j - 2 \) and \( j - 1 \). The interpretation (as in Section 3.2) is that, from the perspective of period \( j - 1 \), generation \( j - 2 \) is the short-horizon clientele whereas generation \( j - 1 \)
is the long-horizon clientele.

**Proposition 7.** If \( \gamma > 1 \), an increase in the endowment of the long-horizon clientele relative to the short-horizon clientele (i.e., an increase in \( \alpha_{j-1} \), holding \( R\alpha_{j-2} + \alpha_{j-1} \) constant) raises the optimal supply \( B^* \) of two-period bonds. The result is reversed if \( \gamma < 1 \).

This type of comparative static is novel and cannot be undertaken in a two-period OLG model since there is only one type of investor at any point in time in such a setup. Practical intuition suggests that the optimal maturity structure should involve more short-term debt when the short-horizon clientele is wealthier. We confirm this intuition when agents’ coefficient of relative risk aversion \( \gamma \) is strictly larger than one. In this case an increase in the clientele for two-period bonds prompts the government to lengthen the maturity structure. As in Proposition 2, the result is reversed when \( \gamma < 1 \), and the clientele mix has no effect on the optimal maturity structure for logarithmic preferences (\( \gamma = 1 \)).

### 5.2 Excess Returns on Long-Term Bonds

The optimal maturity structure implements the first-best allocation. Therefore, the relative returns of one- and two-period bonds under the optimal maturity structure can be determined from the characterization of \( c^*_j,s \) and \( c^*_{j+1,s} \) given in Proposition 4. Recall that the first-order condition of generation \( j-1 \) is

\[
\mathbb{E} \left\{ u'(c_{j+1,s}) \left[ L^2 - RR_s \right] \right\} = 0, \tag{28}
\]

where the term in square brackets is the excess return of two-period bonds relative to the strategy of investing in one-period bonds and rolling over. This excess return is high when \( R_s \) is low, which is also when \( c^*_{j+1,s} \) is low. Therefore, two-period bonds are a valuable hedge from generation \( j-1 \)'s viewpoint, and return less on average than one-period bonds over a two-period horizon:

\[
(L^*)^2 < \mathbb{E} [RR_s]. \tag{29}
\]

To compare returns over a one-period horizon, we use the first-order condition of generation \( j-2 \):

\[
\mathbb{E} \left\{ u'(c_{j,s}) \left[ \frac{L^2}{R_s} - R \right] \right\} = 0. \tag{30}
\]

We know from Proposition 4 that when \( \gamma \) is sufficiently large, generation \( j-2 \) consumes less in state \( L \) in the first-best allocation. Therefore, two-period bonds are a valuable hedge from the viewpoint
of that agent as well and they return less on average than one-period bonds over a one-period horizon:

$$E \left( \frac{(L^*)^2}{R_s} \right) < R. \tag{31}$$

However, this inequality is reversed when $\gamma \leq 1$. Thus, it is possible that two-period bonds outperform on average one-period bonds over a one-period horizon, while underperforming over two periods. Proposition 8 summarizes these results.

**Proposition 8.** Under the optimal maturity structure, two-period bonds earn lower returns on average than one-period bonds over a two-period horizon. They also earn lower returns on average over a one-period horizon if $\gamma$ is sufficiently large, but higher returns if $\gamma \leq 1$.

Proposition 8 suggests that positive excess returns on long-term (real) bonds can be a symptom of an excessive supply of long-term bonds. Indeed, generations present in the market in period $j - 1$ require positive excess returns from long-term bonds if they consume relatively less when interest rates are high. At the same time, generation $j$ consumes more when interest rates are high because it earns a high return on their endowments. Thus, the consumption of current and future generations covaries negatively, implying inefficient risksharing. Risksharing could be made efficient if markets were complete and future generations could trade today. But alternatively, the government can improve risksharing by shortening the maturity structure. Indeed, replacing long-term bonds by short-term bonds raises the consumption of generations $j - 2$ and $j - 1$ when interest rates are high, while also reducing taxes of future generations when interest rates are low.

**5.3 Funding Cost Minimization vs. Welfare Maximization**

Another implication of Proposition 8 is that the government should not necessarily strive to equalize expected returns across bonds of different maturities. Suppose, for example, that expected returns of long-term bonds are below those for short-term bonds. The government could respond by tilting issuance towards long-term bonds as this lowers its expected funding costs. Our model suggests, however, that minimizing expected funding costs should not be the only objective. Indeed, long-term bonds have an implicit cost: taxes on future generations must increase when interest rates are low, which is also when consumption of future generations is low.

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18 Mathematically, (29) can be consistent with the reverse of (31) because of Jensen’s inequality.

19 Our model also implies a downward sloping real yield curve under the optimal government debt policy. Two well documented empirical facts are that excess returns on long-term nominal bonds are positive on average and that the nominal yield curve is upward sloping on average. However the empirical evidence on the real term structure is less clear, see e.g. Ang, Bekaert, and Wei (2008).
However the following proposition suggests that in response to a *shock* to the clientele mix, a welfare maximizing government could *appear* to respond to prices in a way consistent with the objective of minimizing expected funding costs.

**Proposition 9.** If $\gamma > 1$, an increase in the endowment of the long-horizon clientele relative to the short-horizon clientele (i.e., an increase in $\alpha_{j-1}$, holding $R\alpha_{j-2} + \alpha_{j-1}$ constant) lowers the two-period equilibrium yield $L^*$ under the optimal debt policy. The result is reversed if $\gamma < 1$.

Proposition 9 characterizes how the equilibrium yield ($L^*$) on long-term debt under the optimal debt policy responds to a change in the clientele mix, and thus complements Proposition 7, which characterizes the response of the optimal supply of long-term debt ($B^*$) to a change in the clientele mix. Propositions 7 and 9 together imply that shocks to the clientele mix induce negative comovement between $B^*$ and $L^*$ for any $\gamma \neq 1$. For instance, when changes to the clientele mix raise the price of two-period bonds, they also prompt the government to issue more such bonds.

## 6 Conclusion

This paper provides a novel theory of optimal maturity structure based on clienteles. Clienteles arise endogenously because generations differ in their investment horizon. In this setting, the maturity structure of government debt affects welfare: the government can improve intergenerational risksharing by effectively replicating the actions of future generations, thereby alleviating a fundamental limited participation problem. Our model lends itself naturally to the analysis of demand and supply effects on the yield curve. For instance, our model provides a natural explanation of why a lengthening of the maturity structure is typically accompanied by an increase in the slope of the curve. Finally, this framework has implications for the desirability of debt issuance policies that attempt to minimize expected interest costs.

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\[20\] Note that the resulting change in $L^*$ is due to the combined effect of the demand shock and of the induced change in supply.
**APPENDIX**

**Proof of Proposition 1:** We first show that demands are increasing in \( L \). Let \( \omega \equiv R/L^2 \) and \( y_s = 1/R_s \). The first-order condition of generation \( j - 2 \) can be written as

\[
\mathbb{E} \{ u'(R^2 \alpha_{j-2} + (y_s - \omega)B^{j-2}) (y_s - \omega) \} = 0. \tag{A.1}
\]

Differentiating this condition implicitly with respect to \( \omega \), we find

\[
\frac{\partial B^{j-2}}{\partial \omega} = \frac{\mathbb{E} u'[R^2 \alpha_{j-2} + B^{j-2}(y_s - \omega)] + \mathbb{E} [u''(R^2 \alpha_{j-2} + B^{j-2}(y_s - \omega)) B^{j-2}(y_s - \omega)]}{\mathbb{E} [u''(R^2 \alpha_{j-2} + B^{j-2}(y_s - \omega))(y_s - \omega)^2]}.
\]

Since \( u' > 0 \) and \( u'' < 0 \), a sufficient condition for \( \partial B^{j-2}/\partial \omega < 0 \) is

\[
\mathbb{E} [u''(R^2 \alpha_{j-2} + B^{j-2}(y_s - \omega)) B^{j-2}(y_s - \omega)] > 0
\]
\[
\iff \mathbb{E} [F(y_s)u'(R^2 \alpha_{j-2} + B^{j-2}(y_s - \omega)) (y_s - \omega)] > 0, \tag{A.2}
\]

where

\[
F(y) \equiv \frac{u''[R^2 \alpha_{j-2} + B^{j-2}(y - \omega)]}{u'[R^2 \alpha_{j-2} + B^{j-2}(y - \omega)]} B^{j-2}.
\]

Given (A.1), condition (A.2) holds if the function \( F(y) \) is increasing. Because of CRRA utility,

\[
F'(y) = \frac{d}{dy} \left[ -\frac{\gamma B^{j-2}}{R^2 \alpha_{j-2} + B^{j-2}(y - \omega)} \right] = \frac{\gamma(B^{j-2})^2}{[R^2 \alpha_{j-2} + B^{j-2}(y - \omega)]^2} > 0.
\]

Since the sign of \( \partial B^{j-2}/\partial \omega \) is negative, that of \( \partial B^{j-2}/\partial L \) is positive.

The first-order condition of generation \( j - 1 \) can be written as

\[
\mathbb{E} \{ u'(R_s R \alpha_{j-1} + (1 - R_s \omega)B^{j-1}) (1 - R_s \omega) \} = 0. \tag{A.3}
\]

Differentiating this condition implicitly with respect to \( \omega \), we find

\[
\frac{\partial B^{j-1}}{\partial \omega} = \frac{\mathbb{E} [u'(R_s R \alpha_{j-1} + B^{j-1}(1 - R_s \omega)) R_s] + \mathbb{E} [u''(R_s R \alpha_{j-1} + B^{j-1}(1 - R_s \omega)) B^{j-1} R_s (1 - R_s \omega)]}{\mathbb{E} [u''(R_s R \alpha_{j-1} + B^{j-1}(1 - R_s \omega))(1 - R_s \omega)^2]}.
\]
Since \( u' > 0 \) and \( u'' < 0 \), the sign of \( \partial B^{j-1}/\partial \omega \) is negative if

\[
E \left[ u'' \left( R \alpha_j + B^{j-1}(1 - R_{\omega}) \right) B^{j-1} R \left( 1 - R_{\omega} \right) \right] > 0
\]

\( \iff \)

\[
E \left[ (G(R_s)u' \left( R \alpha_j + B^{j-1}(1 - R_{\omega}) \right) (1 - R_{\omega}) \right] > 0,
\]

(A.4)

where

\[
G(R_s) \equiv \frac{u'' \left( R \alpha_j + B^{j-1}(1 - R_{\omega}) \right)}{u' \left( R \alpha_j + B^{j-1}(1 - R_{\omega}) \right)} B^{j-1} R.
\]

Given (A.3), condition (A.4) holds if the function \( G(R_s) \) is decreasing. Because of CRRA utility,

\[
G'(R_s) = \frac{d}{dR_s} \left[ -\frac{\gamma B^{j-1} R_s}{R \alpha_j + B^{j-1}(1 - R_{\omega})} \right] = -\frac{\gamma (B^{j-1})^2}{(R \alpha_j + B^{j-1}(1 - R_{\omega}))^2} < 0.
\]

Since the sign of \( \partial B^{j-1}/\partial \omega \) is negative, that of \( \partial B^{j-1}/\partial L \) is positive.

We next determine the asymptotic behavior of the demands when \( L \) reaches its arbitrage bounds. Since demands are increasing, they converge to a limit when \( L^2 \rightarrow RR_H \). To show that the limit is \( \infty \), we proceed by contradiction. Suppose, for example, that \( B^{j-2} \) has a finite limit.

The first-order condition of generation \( j-2 \), Eq. (6), can be written as

\[
p u' \left[ R^2 \alpha_{j-2} + B^{j-2} \left( \frac{1}{R_H} - \omega \right) \right] \left( \frac{1}{R_H} - \omega \right) + (1-p) u' \left[ R^2 \alpha_{j-2} + B^{j-2} \left( \frac{1}{R_L} - \omega \right) \right] \left( \frac{1}{R_L} - \omega \right) = 0.
\]

(A.5)

The case \( L^2 \rightarrow RR_H \) corresponds to \( \omega \rightarrow \frac{1}{R_H} \). If \( B^{j-2} \) has a finite limit \( \overline{B}^{j-2} \), the limit of (A.5) is

\[
(1-p) u' \left[ R^2 \alpha_{j-2} + \overline{B}^{j-2} \left( \frac{1}{R_L} - \frac{1}{R_H} \right) \right] \left( \frac{1}{R_L} - \frac{1}{R_H} \right) = 0.
\]

Since the left-hand side is positive, we find a contradiction. Therefore, \( B^{j-2} \) converges to \( \infty \) and so does \( B^{j-1} \). A similar argument establishes that \( B^{j-2} \) and \( B^{j-1} \) converge to \( -\infty \) when \( L^2 \rightarrow RR_L \).

Finally, the aggregate demand for two-period bonds \( B^{j-2} + B^{j-1} \) is increasing in \( L \), and its values range from \( -\infty \) to \( \infty \). This implies that the market-clearing equation \( B^{j-2}(L)+B^{j-1}(L) = B \) has a unique solution \( L \).
Proof of Proposition 2: We first show that an increase in the relative endowment of the long-horizon clientele, i.e., an increase in \(\alpha_{j-1}\) holding \(R\alpha_{j-2} + \alpha_{j-1}\) constant, raises the aggregate demand for two-period debt \(B^{j-2} + B^{j-1}\) if \(\gamma > 1\), and lowers it if \(\gamma < 1\). Because of CRRA utility, the quantities
\[
\phi^{j-2} = \frac{B^{j-2}/L^2}{R\alpha_{j-2}},
\]
\[
\phi^{j-1} = \frac{B^{j-1}/L^2}{\alpha_{j-1}},
\]
characterizing the fraction of their wealth that generations \(j - 2\) and \(j - 1\) invest in two-period bonds in period \(j - 1\), are independent of \(\alpha_{j-2}\) and \(\alpha_{j-1}\). Therefore, an increase in \(\alpha_{j-1}\) keeping \(R\alpha_{j-2} + \alpha_{j-1}\) constant raises \(B^{j-2} + B^{j-1}\) if and only if \(\phi^{j-1} > \phi^{j-2}\). To compare \(\phi^{j-1}\) and \(\phi^{j-2}\), we use the first-order conditions of generations \(j - 2\) and \(j - 1\), which imply that
\[
\frac{u'(c_{j,H})}{u'(c_{j,L})} = \frac{R_H}{R_L} \frac{u'(c_{j+1,H})}{u'(c_{j+1,L})}. \tag{A.6}
\]
Substituting in the expressions for \(c_{j,s}\) and \(c_{j+1,s}\) given by (5) and (7) (and rewriting them as functions of \(\phi^{j-2}\) and \(\phi^{j-1}\), respectively), and using the fact that utility is CRRA, (A.6) becomes
\[
1 + \phi^{j-2}(\frac{L^2}{R_H} - 1) = 1 + \phi^{j-1}(\frac{L^2}{R_L} - 1) \left(\frac{R_H}{R_L}\right)^{1-\frac{1}{\gamma}}.
\]
Therefore, the quantity
\[
\frac{1 + \phi^{j-2}(\frac{L^2}{R_H} - 1) + \phi^{j-1}(\frac{L^2}{R_L} - 1) - 1}{1 + \phi^{j-2}(\frac{L^2}{R_H} - 1) + \phi^{j-1}(\frac{L^2}{R_L} - 1)} = \frac{(\phi^{j-1} - \phi^{j-2})(\frac{L^2}{R_L} - \frac{L^2}{R_H})}{(1 + \phi^{j-2}(\frac{L^2}{R_H} - 1))(1 + \phi^{j-1}(\frac{L^2}{R_L} - 1))}
\]
has the same sign as
\[
\left(\frac{R_H}{R_L}\right)^{1-\frac{1}{\gamma}} - 1.
\]
Therefore, the sign of \(\phi^{j-1} - \phi^{j-2}\) is the same as the sign of \(\gamma - 1\).
The impact of a change in the relative endowments of the two clienteles on the two-period yield follows by differentiation of the market clearing condition $B^j - 2(L) + B^j - 1(L) = B$:

$$\frac{dL}{d\alpha_i} \bigg|_{d(R\alpha_{i-1})=-d\alpha_{i-1}} = \frac{L^2}{\frac{\partial B^i}{\partial L} + \frac{\partial B^{i-1}}{\partial L}} \left( \phi^j - 2 - \phi^j - 1 \right). \quad (A.7)$$

This has the same sign as $1 - \gamma$. Hence an increase in the relative wealth if the long-horizon clientele in period $j - 1$ lowers the two-period interest rate if $\gamma > 1$, and the result is reversed if $\gamma < 1$. □

**Proof of Proposition 3:** The market clearing condition for two-period bonds is $B^j - 2(L) + B^j - 1(L) = B$. The implicit function theorem implies

$$\frac{dL}{dB} = \frac{1}{\frac{\partial B^i}{\partial L} + \frac{\partial B^{i-1}}{\partial L}} > 0, \quad (A.8)$$

where the inequality follows from the fact that demands are strictly increasing in $L$. □

**Proof of Lemma 1:** Consider the social planner problem $P$ defined in Section 4.1. First-order optimality implies that

$$\lambda_0 u'(c_{j,s}) = \lambda_i R^{i-1} u'(c_{j+i,s}), \quad \forall i \geq 1, \quad \forall s = H, L. \quad (A.9)$$

With CRRA preferences, this implies

$$c_{j+i,s} = \left( \frac{\lambda_0}{\lambda_i R^{i-1}} \right)^{-\frac{1}{\gamma}} c_{j,s}, \quad \forall i \geq 1, \quad \forall s = H, L. \quad (A.10)$$

Hence the intertemporal resource constraint (14) can be written

$$\left[ 1 + R_{s}^{\frac{1}{\gamma} - 1} \sum_{i=1}^{\infty} \left( \frac{1}{R^{i-1}} \right)^{1-\frac{1}{\gamma}} \left( \frac{\lambda_0}{\lambda_i} \right)^{-\frac{1}{\gamma}} \right] c_{j,s} = e_{j,s}. \quad (A.11)$$

This gives Eq. (15). (A.9) also implies that

$$c_{j+i,s} = \left( \frac{\lambda_1}{\lambda_i R^{i-1}} \right)^{-\frac{1}{\gamma}} c_{j+1,s}, \quad \forall i \geq 2, \quad \forall s = H, L. \quad (A.12)$$
Hence the intertemporal resource constraint in period \( j + 1 \) can be written

\[
\sum_{i=1}^{\infty} \frac{c_{j+i,s}}{R^{i-1}} = \left[ 1 + R^{\frac{j-1}{2}} \sum_{i=1}^{\infty} \left( \frac{1}{R^{i-1}} \right)^{\frac{j-1}{2}} \left( \frac{\lambda_1}{\lambda_{1+i}} \right)^{-\frac{j-1}{2}} \right] c_{j+1,s} = e_{j+1,s},
\]

(A.13)

where \( e_{j+1,s} \equiv R_s(e_{j,s} - c_{j,s}) \). This gives Eq. (16). Finally, (17) follows directly from (A.9).

**Proof of Lemma 2:** First, we prove uniqueness of the complete markets equilibrium. This follows from the fact that the demands \( \{ B^t : t \geq j - 2 \} \) are strictly increasing in \( L \), converge to \(-\infty\) when \( L^2 \) goes to \( RR_L \), and converge to \( \infty \) when \( L^2 \) goes to \( RR_H \). These properties have been proved for \( B^{j-2} \) and \( B^{j-1} \) in the proof of Proposition 1. We now prove them for \( B^j \) and for \( B^t, t \geq j + 1 \).

The first-order condition of generation \( j \) can be written as

\[
E \left\{ u' \left[ R_s R \alpha_j + (1 - R_s \omega) B^j \right] (1 - R_s \omega) \right\} = 0,
\]

(A.14)

where \( \omega \equiv R/L^2 \). Differentiating this condition implicitly with respect to \( \omega \), we find

\[
\frac{\partial B^j}{\partial \omega} = \frac{E \left[ u' \left[ R_s R \alpha_j + B^j(1 - R_s \omega) \right] R_s \right] + E \left[ u'' \left[ R_s R \alpha_j + B^j(1 - R_s \omega) \right] B^j R_s (1 - R_s \omega) \right]}{E \left[ u'' \left[ R_s R \alpha_j + B^j(1 - R_s \omega) \right] (1 - R_s \omega)^2 \right]}.
\]

Since \( u' > 0 \) and \( u'' < 0 \), the sign of \( \partial B^j/\partial \omega \) is negative if

\[
E \left[ u'' \left[ R_s R \alpha_j + B^j(1 - R_s \omega) \right] B^j R_s (1 - R_s \omega) \right] > 0.
\]

(A.15)

An argument analogous to the one made in the proof of Proposition 1 to prove that \( \partial B^{j-1}/\partial L > 0 \) allows to conclude that \( \partial B^j/\partial L \) is positive. Next, the first-order condition of all generation \( t \geq j + 1 \) can be written as

\[
E \left\{ u' \left( R^2 \alpha_t + R^{t+2-(j+1)}(1 - R_s \omega) B^t \right) (1 - R_s \omega) \right\} = 0.
\]

(A.16)

Again, differentiating this condition implicitly with respect to \( \omega \), an argument analogous to the one used in the proof of Proposition 1 for generation \( j - 1 \) allows to conclude that \( \partial B^t/\partial L \) is positive. Finally, the asymptotic behavior of the demands when \( L \) reaches its arbitrage bounds can also be derived along the lines of the proof of Proposition 1. \( \square \)
Now we derive the weights $\beta_j$ and $\beta_{j+1}$ supporting the complete markets equilibrium allocation. Multiplying (5) by $u'(c_j)$ and taking expectations, together with the first-order condition (6), gives

$$
E[u'(c_j)c_j] = R^2 \alpha_{j-2} E[u'(c_j)].
$$

(A.17)

This is Eq. (19). In a similar fashion, multiplying (7) by $u'(c_{j+1})$ and taking expectations, together with the first-order condition (8), gives

$$
E[u'(c_{j+1})c_{j+1}] = \alpha_j L^2 E[u'(c_{j+1})].
$$

(A.18)

Together with (23), this proves Eq. (20).

We now turn to the determination of $(\lambda_{k-1}/\lambda_k R)^{\frac{-1}{k}} \equiv \kappa_{j+k}$ for $k \geq 2$. Multiplying (10) by $u'(c_{j+2})$ and taking expectations, together with the first-order condition (A.14), gives

$$
E[u'(c_{j+2})c_{j+2}] = \alpha_j L^2 E[u'(c_{j+2})].
$$

(A.19)

Eqs. (20) and (A.19) imply that $\kappa_{j+2} = \alpha_j/\alpha_{j-1}$. Next, multiplying (11) for $t = j + 1$ by $u'(c_{j+3})$ and taking expectations, together with the first-order condition for generation $j + 1$, gives

$$
E[u'(c_{j+3})c_{j+3}] = \alpha_{j+1} R^2 E[u'(c_{j+3})].
$$

(A.20)

Eqs. (A.19) and (A.20) imply that $\kappa_{j+3} = \frac{R^2 \alpha_{j+1}}{\alpha_j}$. Together with (23), this proves (22). Finally, multiplying (11) by $u'(c_{t+2})$ for $t \geq j + 2$ and taking expectations, together with the first-order condition for generations $j + 2$ onwards, gives

$$
E[u'(c_{t+2})c_{t+2}] = \alpha_t R^2 E[u'(c_{t+2})], \quad t \geq j + 2.
$$

(A.21)

Eqs. (A.20) and (A.21) imply that $\kappa_{t+1} = \alpha_{t-1}/\alpha_{t-2}$ for all $t \geq j + 3$.

Proof of Proposition 4: Consider the social planner’s representation of the complete markets allocation. First, we show that, for all $t \geq j + 1$, $c_{t,H} > c_{t,L}$. We proceed by contradiction. Suppose there exists $T \geq j + 1$ such that $c_{T,H} \leq c_{T,L}$. Then from (A.9), $c_{t,H} \leq c_{t,L}$ for all $t \geq j$. In particular
this implies $c_{j,H} \leq c_{j,L}$ and

$$
\sum_{i=0}^{\infty} \frac{1}{R^i} (c_{j+1+i,H} - c_{j+1+i,L}) \leq 0.
$$
(A.22)

At the same time, Eq. (14) implies

$$
\sum_{i=0}^{\infty} \frac{1}{R^i} (c_{j+1+i,H} - c_{j+1+i,L}) = R_H e_{j,H} - R_L e_{j,L} - (c_{j,H} R_H - c_{j,L} R_L) \\
\geq (R_H - R_L) [\alpha_j - 2 R^2 + \alpha_{j-1} R + \alpha_j - c_{j,H}],
$$
(A.23)

where the inequality follows from $c_{j,H} \leq c_{j,L}$. Furthermore, absence of arbitrage requires

$$
c_{j,H} \leq \alpha_j - 2 R^2 \leq c_{j,L}.
$$
(A.25)

Therefore

$$
\sum_{i=0}^{\infty} \frac{1}{R^i} (c_{j+1+i,H} - c_{j+1+i,L}) \geq (R_H - R_L) [\alpha_{j-1} R + \alpha_j] > 0.
$$
(A.26)

This contradicts (A.22). Therefore, $c_{t,H} > c_{t,L}$ for all $t \geq j + 1$.

Now, we analyze the state-dependence of $c_j$. Using (A.10) together with the fact that $c_{t,H} > c_{t,L}$ for all $t \geq j + 1$ implies

$$
\sum_{i=0}^{\infty} \frac{1}{R^i} (c_{j+1+i,H} - c_{j+1+i,L}) = \sum_{i=0}^{\infty} \frac{1}{R^i} \left( (\Lambda_{i+1,H})^{-1} c_{j,H} - (\Lambda_{i+1,L})^{-1} c_{j,L} \right) > 0,
$$
(A.27)

where $\Lambda_{i+1,s} \equiv \frac{\Lambda_0}{\lambda_{i+1} R^s R^i}$. In the limit as $\gamma \to \infty$, this implies $c_{j,H} > c_{j,L}$. For the case $\gamma \leq 1$, we use Eq. (A.11). The RHS of this equation is decreasing in $R_s$ (by definition of $e_{j,s}$) while the term in square brackets on the LHS is increasing in $R_s$, which implies $c_{j,H} < c_{j,L}$.

**Proof of Proposition 5:** A proof by construction is given in the text of Section 4.2.
Proof of Proposition 6: Let $V_{j+2,s}$ denote the present value of $c_{j+2}$, $c_{j+3}$, etc. in state $s$ as of period $j + 2$. By aggregating budget constraints of generations $j$ onwards, we obtain

$$V_{j+2,s} = RR_s \left\{ \alpha_j + \frac{1}{R_s} \sum_{i=1}^{\infty} \frac{\alpha_{j+i}}{R^{i-1}} \right\} - D_{j,s} \right\} \right\} = R \left[ R_s \alpha_j + A_{j+1} - \left( 1 - \frac{RR_s}{L^2} \right) B \right],$$

(A.28)

where the second equality follows by defining

$$A_{j+1} \equiv \sum_{i=2}^{\infty} \frac{\alpha_{j+i}}{R^i},$$

(A.30)

and from the fact that $D_{j,s} = \left( -\frac{R}{L^2} + \frac{1}{R} \right) B$. Multiplying $V_{j+2,s}$ by $u'(c_{j+1,s})$ and taking expectations, we get

$$E[u'(c_{j+1,s})V_{j+2,s}] = (L^2 \alpha_j + RA_{j+1})E[u'(c_{j+1,s})].$$

(A.31)

Therefore, taking the ratio of (A.31) and (20), we obtain

$$\frac{E[u'(c_{j+1,s})V_{j+2,s}]}{E[u'(c_{j+1,s})c_{j+1,s}]} = \frac{L^2 \alpha_j + RA_{j+1}}{L^2 \alpha_{j-1}}.$$

(A.32)

Now, using (A.12), we can write

$$V^*_{j+2,s} \equiv \sum_{i=2}^{\infty} \frac{c^*_{j+i,s}}{R^{i-2}} = \left[ \sum_{i=2}^{\infty} \frac{1}{R^{i-2}} \left( \frac{\lambda_1}{\lambda_i R^{i-1}} \right)^{-\frac{1}{\gamma}} \right] c^*_{j+1,s} = \left( \beta_{j+1} R \right)^{-\frac{1}{\gamma}} c^*_{j+1,s}, \quad s = H, L,$$

(A.33)

where $\beta_{j+1}$ can be interpreted as the weight of generations $j + 2$, $j + 3$ and so forth relative to generation $j + 1$. Now we use (7) and (A.29) to identify the components on each side of Eq. (A.33) that are independent of $R_s$. Equating these components gives

$$B^{*j-1} = \left( \beta_{j+1} R \right)^{-\frac{1}{\gamma}} R(A_{j+1} - B^*),$$

(A.34)

Indeed, Eq. (A.33) can be read as $(c^*_{j+1,s})^{-\gamma} = \beta_{j+1} R(V^*_{j+2,s})^{-\gamma}$, for all $s$.\footnote{Indeed, Eq. (A.33) can be read as $(c^*_{j+1,s})^{-\gamma} = \beta_{j+1} R(V^*_{j+2,s})^{-\gamma}$, for all $s$.}
and using the market-clearing condition \( B^{j-2}(L) + B^{j-1}(L) = B \), we get

\[
B^* = \frac{R}{(\beta_{j+1} R)^{\gamma}} A_{j+1} + B^{*j-2} \left( \frac{R}{1 + (\beta_{j+1} R)^{\gamma}} \right).
\] (A.35)

This allows us to characterize the size of the optimal supply of two-period debt. When \( \gamma \leq 1 \), Proposition 4 together with (5)-(7) imply \( B^{*j-2} > 0 \), hence \( B^* > 0 \). By contrast, when \( \gamma \to \infty \), Proposition 4 implies \( B^{*j-2} < 0 \). When \( \alpha_{j-2} \) goes to zero (so that the long horizon clientele prevails), \( B^{*j-2} \) vanishes and \( B^* > 0 \). When \( \alpha_{j-1} \) goes to zero (so that the short horizon clientele prevails), \( (\mu R)^{\frac{1}{\gamma}} \) tends to infinity and therefore \( B^* < 0 \).

**Proof of Proposition 7:** Consider the complete markets equilibrium defined in Section 4.1. By the same reasoning as in the proof of Proposition 2, increasing the endowment of generation \( j-1 \) relative to generation \( j-2 \) causes an increase in the aggregate demand \( \tilde{B}^{j-2} + \tilde{B}^{j-1} \) of generations \( j-2 \) and \( j-1 \) iff \( \gamma > 1 \). Hence, (24) implies that increasing the endowment of generation \( j-1 \) relative to generation \( j-2 \) causes an increase in \( B^* \) iff \( \gamma > 1 \).

**Proof of Proposition 8:** From (28), the excess expected returns of two-period bonds over two periods is

\[
L^2 - \mathbb{E}[RR_s] = \frac{R}{\mathbb{E}[u'(c_{j+1})]} \text{cov}(u'(c_{j+1}), R_s).
\] (A.36)

From (30), the excess expected returns of two-period bonds over one period is

\[
\mathbb{E} \left[ \frac{L^2}{R_s} \right] - R = -\frac{L^2}{\mathbb{E}[u'(c_j)]} \text{cov} \left( u'(c_j), \frac{1}{R_s} \right).
\] (A.37)

The sign of these excess returns under the optimal debt issuance policy follows from Proposition 4 which pins down the sign of the covariance terms.

**Proof of Proposition 9:** The two-period rate prevailing under the optimal maturity structure coincides with the two-period rate \( L^* \) prevailing in the complete markets equilibrium of Section 4.1 defined in (23). We have shown in the proof of Proposition 7 that increasing the endowment of generation \( j-1 \) relative to generation \( j-2 \) causes an increase in the aggregate demand \( \tilde{B}^{j-2} + \tilde{B}^{j-1} \) of generations \( j-2 \) and \( j-1 \) iff \( \gamma > 1 \). At the same time such a change in endowments has no direct effect on the demand of later generations. Since all generations have a demand for two-period bonds
that is increasing in $L$, $L$ has to decrease to restore the complete markets equilibrium (i.e., for the market clearing condition (12) to hold) iff $\gamma > 1$. Hence, increasing the endowment of generation $j - 1$ relative to generation $j - 2$ causes a fall in $L^*$ iff $\gamma > 1$. 

\[ \Box \]
References


